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Functional representation for fermionic quantum fields

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A functional representation for fermionic quantum fields is developed in analogy to familiar results for bosonic fields. The infinite Clifford algebra of the field anticommutator is realized reducibly on a Grassmann functional space. On this space, transformation groups may be represented without normal ordering with respect to a Fock vacuum, and a projective representation for the two-dimensional conformal group is found, which is compared to the corresponding representation in terms of bosonic fields. When a quadratic Hamiltonian for the Fermi fields is posited, a Fock space can be constructed after a prescription for filling the Dirac sea is selected. Different filling prescriptions lead to inequivalent Fock spaces within the functional space. Explicit eigenfunctionals exhibit the peculiarities of fermionic field theory, such as fractional charge, Berry's phase, and anomalies.

I. INTRODUCTION

The Schrödinger picture is frequently used to give an explicit representation for various phenomena that arise in a bosonic quantum field theory. One considers functionals of the dynamical variable—the field at fixed time—and realizes the canonical momentum operator by functional differentiation. Composition is achieved by functional integration, and a natural inner product may be defined. In this way one arrives at explicit wave functionals, which describe effects of interest either exactly or approximately, and physical/mathematical intuition derived from ordinary quantum mechanics may be used to advance understanding.

An important advantage of this approach is the ability to discuss kinematical topics in representation theory, without reference to dynamics. This is to be contrasted with the usual field-theoretic procedure where the need to regularize and renormalize kinematical objects, such as generators of transformations, brings in dynamics through the normal-ordering algorithm (or equivalently through the operator-product expansion), which is defined with reference to a Fock vacuum, i.e., to the lowest eigenstate of some quadratic Hamiltonian. As we shall review below, within the Schrödinger picture one may regularize and renormalize intrinsically, without reference to any vacuum state. At the very least this is a stylistic advantage—quantum representations in field theory should be achieved independently of dynamics, just as in quantum mechanics. Moreover, when there is no well-defined notion of a Fock vacuum, as in de Sitter space, the intrinsic method is the only one available.

While the Schrödinger picture for a bosonic quantum field theory is well understood,[1] the analog for a fermionic field theory has not been developed, and is presented in this paper.[2]

In Sec. II we review and enlarge upon our previous work on the functional Schrödinger picture for bosonic quantum field theory,[1] while our approach to models with fermions comprises Sec. III. Since the functional representation that we introduce for fermions is new and unfamiliar, we illustrate it in an especially transparent example: "field theory" on a finite number of spatial points, i.e., fermionic quantum mechanics.

Although our main purpose is to represent functionally fermion (anti)commutators without reference to dynamics, we also describe within our formalism fermion dynamics governed by quadratic Hamiltonians. For these, the eigenfunctionals may be constructed and they explicitly exhibit the field-theoretic peculiarities of fermions: the need to define a Dirac sea, fractional charge, anomalies, and Berry's phase. Some further explanatory remarks are relegated to the Appendix.

II. BOSONIC FIELD THEORIES

A. The function space

We consider at fixed time a space of functionals of $\phi(\mathbf{x})$, and view them as kets: $|\Psi\rangle \leftrightarrow \Psi(\phi)$. An inner product is defined by functional integration,

$$\langle \Psi_1 | \Psi_2 \rangle = \int \mathcal{D}\phi \, \Psi_1^*(\phi)\Psi_2(\phi) \;, \tag{2.1}$$

so that the dual space of bras consist of complex-conjugated functionals: $\langle \Psi | \leftrightarrow \Psi^*(\phi)$. Operators are represented by functional kernels:

$$\mathcal{O} | \Psi \rangle \leftrightarrow \int \mathcal{D}\tilde{\phi} \, \mathcal{O}(\phi,\tilde{\phi})\Psi(\tilde{\phi}) \;. \tag{2.2}$$

The field operator $\Phi(\mathbf{x})$ is represented by a diagonal kernel $\phi(\mathbf{x})\delta(\phi-\tilde{\phi})$, the canonical momentum operator $\Pi(\mathbf{x})$ by $(1/i)[\delta/\delta\phi(\mathbf{x})]\delta(\phi-\tilde{\phi})$. Evidently, the former acts on functionals of ϕ by multiplication, the latter by (functional) differentiation.

Fock bases in this space and in the dual space are readily constructed. The *Fock vacuum* $|\Omega\rangle$ is represent-

ed by a Gaussian functional with specific covariance Ω, which is symmetric, can be complex, but possesses a positive-definite real part Ω_R:

$$|\,\Omega\rangle \leftrightarrow \det{}^{1/4}\left[\frac{\Omega_R}{\pi}\right]\exp\left[-\frac{1}{2}\int \phi\Omega\phi\right],$$
$$\Omega=\Omega_R+i\Omega_I,\quad \Omega(\mathbf{x},\mathbf{y})=\Omega(\mathbf{y},\mathbf{x})\;; \tag{2.3a}$$

$$\langle\Omega\,| \leftrightarrow \det{}^{1/4}\left[\frac{\Omega_R}{\pi}\right]\exp\left[-\tfrac{1}{2}\int \phi\Omega^*\phi\right]. \tag{2.3b}$$

[An obvious functional notation is used throughout:

$$\int \phi\Omega\phi \equiv \int d\mathbf{x}\,d\mathbf{y}\,\phi(\mathbf{x})\Omega(\mathbf{x},\mathbf{y})\phi(\mathbf{y})$$

and the determinant is functional.] The reason for the nomenclature is that the above Fock vacuum is annihilated by an operator A that is linear in Φ and Π:

$$A=\frac{1}{\sqrt{2}}\int \Omega_R^{-1/2}\left[\Omega\phi+\frac{\delta}{\delta\phi}\right], \tag{2.4}$$

$$A\,|\,\Omega\rangle=0\,. \tag{2.5}$$

A satisfies the annihilation-creation commutation relations:

$$[A(\mathbf{x}),A^\dagger(\mathbf{y})]=\delta(\mathbf{x}-\mathbf{y})\,,$$
$$[A(\mathbf{x}),A(\mathbf{y})]=0\,. \tag{2.6}$$

Higher basis states are polynomials in ϕ multiplying $|\,\Omega\rangle$, and they are orthonormalized if linear combinations corresponding to "functional Hermite polynomials" are taken. This defines a *Fock space* within our function space.

In our function space, different Fock bases constructed with different covariances can be inequivalent. This happens because field theory possesses an infinite number of degrees of freedom. For consider two Fock vacua with covariances Ω_1 and Ω_2. Their overlap is

$$e^{-N}\equiv\langle\Omega_1\,|\,\Omega_2\rangle\,, \tag{2.7a}$$

$$N=\tfrac{1}{2}\,\mathrm{tr}\ln\frac{\Omega_1+\Omega_2^*}{2(\Omega_{1R}\Omega_{2R})^{1/2}}\,. \tag{2.7b}$$

For example, when the covariances are real and translation invariant (i.e., the dependence on $\mathbf{x}$ and $\mathbf{y}$ is through the difference and the covariance kernel may be diagonalized by Fourier transformation) N is given by

$$N=\tfrac{1}{2}V\int_{\mathbf{k}}\ln\tfrac{1}{2}\left[\left[\frac{\Omega_1(\mathbf{k})}{\Omega_2(\mathbf{k})}\right]^{1/2}+\left[\frac{\Omega_2(\mathbf{k})}{\Omega_1(\mathbf{k})}\right]^{1/2}\right]. \tag{2.8}$$

Here $\Omega_i(\mathbf{k})$ are the Fourier-transformed kernels and V is the volume of space. Even ignoring the (infrared) infinity associated in the spatial volume, N/V will still diverge in the ultraviolet unless Ω_1 and Ω_2 approach each other rapidly at large $\mathbf{k}$. Since the integrand is positive, the divergence sets e^{-N} to zero, and the overlap (2.7a) vanishes. Then also all higher states built on $|\,\Omega_1\rangle$ and $|\,\Omega_2\rangle$, respectively, are mutually orthogonal.

The existence of inequivalent Fock bases gives rise to inequivalent representations for operators that are normal ordered with respect to different bases. As will be seen below, this means in practical terms that results depend on the covariance Ω. An equivalent statement is that given any normal-ordering prescription one may always perform a Bogoliubov transformation, which changes results because it is not unitarily implementable, owing to the infinite number of degrees of freedom.

However, calculations can also be performed directly on the function space, without choosing a Fock basis. In this case one is dealing with a "field basis" consisting of field eigenstates $|\,\phi\rangle$ with functional δ-function normalization:

$$\langle\phi_1\,|\,\phi_2\rangle=\delta(\phi_1-\phi_2)$$
$$=\int \mathcal{D}\alpha\,\exp\left[i\int_{\mathbf{x}}\alpha(\mathbf{x})[\phi_1(\mathbf{x})-\phi_2(\mathbf{x})]\right]. \tag{2.9}$$

The functionals $\Psi(\phi)$ may be viewed as overlaps,

$$\langle\phi\,|\,\Psi\rangle=\Psi(\phi)\,, \tag{2.10}$$

and the functional kernels $\mathcal{O}(\phi,\tilde{\phi})$ as matrix elements:

$$\langle\phi\,|\,\mathcal{O}\,|\,\tilde{\phi}\rangle=\mathcal{O}(\phi,\tilde{\phi})\,. \tag{2.11}$$

Provided the functional kernels can be well defined, one gets results that make no reference to any preselected Fock basis; i.e., they are independent of Ω.

When specific dynamics is in mind, and a specific Hamiltonian is given, it may be that a unique Fock vacuum is determined by the quadratic part of the Hamiltonian. In this case a "natural" choice for the covariance, for the normal-ordering convention, is at hand, and one of the inequivalent Fock spaces is selected. But it seems preferable to use a framework which yields well-defined and unique representations without preselecting dynamics or a Fock vacuum. Moreover, there certainly exist Hamiltonians for which the concept of ground state is inapplicable. Examples are the Liouville theory, which does not possess a lowest-energy eigenstate owing to the exponential interaction,[3] and field theory in de Sitter space, where the Hamiltonian is time dependent.[4]

B. Representing transformations in bosonic quantnm field theory

In a canonical, fixed-time framework for field theory, a Lie group of transformations acting on fields Φ can be represented infinitesimally by constructing generators Q, which typically are polynomials in Φ and Π. The generators effect the field transformation and generically follow the Lie algebra of the group:

$$[Q_1,Q_2]=iQ_{(1,2)}\,. \tag{2.12}$$

Equation (2.12) holds in classical field theory, with Poisson bracketing. It appears to hold in quantum theory with canonical commutation; but this is misleading since Q is an ill-defined quantum-mechanical operator containing products of operators at the same point.

To arrive at well-defined generators, a three-step procedure is adopted. First, the formal expression for Q is regulated in some fashion so that no ill-defined products occur: $Q\to Q^R$. Second, the singular portions of Q^R,

which are ill defined in the absence of the regularization, are isolated and removed. For the simple models that we consider, a c-number subtraction q^R suffices. Finally, third, the regulators are removed from the subtracted expression, leaving well-defined generators, which we denote by $:Q:$, even though the colons do not necessarily signify normal ordering:

$$:Q:\equiv\lim_R (Q^R-q^R) . \tag{2.13}$$

The generators, $:Q:$, well defined in the above manner, continue to generate the infinitesimal transformations on the canonical variables. However, nonlinear relations such as (2.12) can be modified. From (2.13) one gets

$$[:Q_1:,:Q_2:]=i:Q_{(1,2)}:+i\lim_R q^R_{(1,2)} . \tag{2.14}$$

If the limit of $q^R_{(1,2)}$ is nonzero, the quantum-field-theoretic realization of the Lie algebra acquires an extension, not seen in the classical theory—this is the origin of an anomaly.

It still remains to decide how the regularizing subtraction q^R should be determined. In the conventional approach, a Fock vacuum is chosen, q^R is the expectation of Q^R in that state and $:Q:$ is normal ordered with respect to that vacuum. While this procedure may produce well-defined results, they depend on the covariance of the vacuum; specifically, the extension in the Lie algebra can depend on the vacuum. An equivalent statement is that a Bogoliubov transformation changes the normal ordering prescription and can modify the extension. In this way, one is led to inequivalent representations.[5]

We propose that the subtraction be carried out in terms of field states. Because the representation of the regulated generator

$$\langle\phi_1|Q^R(\Phi,\Pi)|\phi_2\rangle=Q^R\left[\phi_1,\frac{1}{i}\frac{\delta}{\delta\phi_1}\right]\delta(\phi_1-\phi_2) \tag{2.15}$$

is a functional distribution involving a functional δ function, expression (2.15) does not yield useful information about the singularities of Q^R when the regulators are removed. However, one may also consider the functional representation kernel for the finite transformation,

$$U^R(\phi_1,\phi_2)=\langle\phi_1|e^{-i\tau Q^R}|\phi_2\rangle \tag{2.16}$$

which implements the (regulated) transformation on states:

$$e^{-i\tau Q^R}|\Psi\rangle\leftrightarrow\int \mathcal{D}\tilde\phi\, U^R(\phi,\tilde\phi)\Psi(\tilde\phi) . \tag{2.17}$$

Evidently, U^R satisfies a functional Schrödinger-type equation:

$$i\frac{\partial}{\partial\tau}U^R(\phi_1,\phi_2)=Q^R\left[\phi_1,\frac{\delta}{i\delta\phi_1}\right]U^R(\phi_1,\phi_2) ,$$
$$U^R(\phi_1,\phi_2)|_{\tau=0}=\delta(\phi_1-\phi_2) . \tag{2.18}$$

$U^R(\phi_1,\phi_2)$ is a functional of ϕ_1 and ϕ_2 [rather than a functional distribution], with specific dependence on ϕ_1 and ϕ_2, as well as on the regulators. So its behavior when the regulators are removed may be explicitly studied.

Generically, U^R becomes singular in the absence of regulators, but in the simple models that we consider, the infinities are confined to a ϕ-independent phase, $e^{-i\tau q^R}$. Thus $e^{i\tau q^R}U^R(\phi_1,\phi_2)$ possesses a well-defined limit and provides a representation for the finite transformation. Since

$$e^{i\tau q^R}U^R(\phi_1,\phi_2)=\langle\phi_1|e^{-i\tau(Q^R-q^R)}|\phi_2\rangle \tag{2.19}$$

the regularizing subtraction for the generator is q^R, which in our approach is determined intrinsically, without referring to any Fock space or preselecting any vacuum. Note finally that when $q^R_{(1,2)}$ survives in the limit that the regulators are removed, the above construction yields a projective representation for the transformation group, with an explicitly determined two-cocycle whose infinitesimal form, $\lim_R q^R_{(1,2)}$, is a Schwinger term in the generator algebra.

C. Two-dimensional conformal transformations

1. *Intrinsic method*

Quantum-field-theoretic representation for conformal transformations on two-dimensional space-time illustrates well our program, and we review and expand our analysis of this problem.[1] Conformal transformations in two dimensions form a doubly infinite transformation group, whereby $x\pm t$ are taken into arbitrary, and in general, different functions of $x\pm t$. At fixed time, the infinitesimal transformation law for the coordinate x, $\delta_f x=-f(x)$, obeys a Lie algebra given by the Lie brackets:

$$[\delta_f,\delta_g]x=-\delta_{(f,g)}x , \tag{2.20a}$$

$$(f,g)=fg'-gf' . \tag{2.20b}$$

To obtain a representation in terms of quantized boson fields, we consider a field operator $\chi(x)$ that satisfies the [equal-time] commutation relation

$$[\chi(x),\chi(y)]=i\delta'(x-y)\equiv k(x,y)$$
$$=\int\frac{dp}{2\pi}e^{-ip(x-y)}p . \tag{2.21}$$

[One may think of χ as $(1/\sqrt{2})(\Pi+\Phi')$, where Π and Φ are canonically conjugate and the dash signifies differentiation with respect to argument; but this is not necessary.] The formal generator of the transformation

$$Q_f=\tfrac{1}{2}\int dx\,\chi(x)f(x)\chi(x) \tag{2.22}$$

transforms the field operator χ as

$$\delta_f\chi=i[Q_f,\chi]=(f\chi)' . \tag{2.23}$$

The generators follow the Lie algebra (2.20),

$$[Q_f,Q_g]=iQ_{(f,g)} \tag{2.24}$$

when the commutator is evaluated formally, without care about the product of χ with itself at the same point.

To regulate the generator, we promote f to a bilocal function $F(x,y)$ and define,

$$Q^R \equiv Q_F = \tfrac{1}{2} \int \chi F \chi \ , \tag{2.25}$$

while removing the regulator consists of passing to the local limit:

$$F(x,y) \to \tfrac{1}{2}[f(x)+f(y)]\delta(x-y) \ . \tag{2.26}$$

$F(x,y)$ is taken to be real and symmetric in (x,y) and sufficiently well behaved near $x \approx y$ to permit all formal manipulations. Of course, Q_F no longer generates conformal transformations, rather general linear canonical transformations.

In the field representation for the above quantities, we represent χ by

$$\langle \phi_1 | \chi(x) | \phi_2 \rangle = \frac{1}{\sqrt{2}} \left[\frac{1}{i} \frac{\delta}{\delta \phi_1(x)} + \phi_1'(x) \right] \delta(\phi_1 - \phi_2) \ . \tag{2.27}$$

The regulated transformation kernel $U^R(\phi_1,\phi_2) \equiv U(\phi_1,\phi_2;F) = \langle \phi_1 | e^{-iQ_F} | \phi_2 \rangle$ may be found; it is a Gaussian times a normalization factor N_F (Ref. 1):

$$U(\phi_1,\phi_2;F) = N_F \exp\left[-\int \phi_1 k \phi_2 \right] \exp\left[\frac{i}{2} \int (\phi_1-\phi_2) K_F (\phi_1 - \phi_2) \right] , \tag{2.28}$$

$$N_F = \det{}^{-1/2} F^{1/2} \left[\frac{2\pi i}{\mathcal{F}} \sin \frac{\mathcal{F}}{2} \right] F^{1/2} \ , \tag{2.29}$$

$$K_F = F^{-1/2} \left[\mathcal{F} \cot \frac{\mathcal{F}}{2} \right] F^{-1/2} \ , \tag{2.30}$$

$$\mathcal{F} \equiv F^{1/2} k F^{1/2} \ . \tag{2.31}$$

In the local limit, K_F attains a well-defined expression

$$K_F(x,y) \to K_f(x,y) = \frac{1}{f(x)} \left[\int \frac{d\lambda}{2\pi} (\lambda \cot \tfrac{1}{2}\lambda) \exp\left[-i\lambda \int_y^x \frac{dz}{f(z)} \right] \right] \frac{1}{f(y)}$$

$$= -i\pi \frac{1}{f(x)} \left[P \csc^2 \pi \int_y^x \frac{dz}{f(z)} \right] \frac{1}{f(y)} \ . \tag{2.32}$$

(P means principal value.) The normalization constant N_F, however, diverges. The divergence resides in an unimportant constant factor Z (which may be removed by redefining the measure of functional integration) and in a phase e^{iq_F}, which is determined in imaginary $\tau (\tau F \to -i\tau F)$:

$$q_F = \tfrac{1}{4} \operatorname{tr} F\omega \ , \tag{2.33}$$

$$\omega(x,y) \equiv |k|(x,y) = \int \frac{dp}{2\pi} e^{-ip(x-y)} |p|$$

$$= -P \frac{1}{\pi (x-y)^2} \ . \tag{2.34}$$

[We use the notation $|\cdots|$ on a kernel to represent the absolute-value kernel, defined through its spectral representation by taking the absolute value of the eigenvalues, as in (2.21) and (2.34).] It follows that

$$Z^{-1} e^{iq_F} U(\phi_1,\phi_2;F) = Z^{-1/2} \langle \phi_1 | e^{-i(Q_F - q_F)} | \phi_2 \rangle Z^{-1/2} \tag{2.35}$$

possesses a well-defined local limit, and we are led to define a renormalized generator

$$:Q_f: \equiv \lim_{F \to f} (Q_F - \tfrac{1}{4} \operatorname{tr} F\omega) \ . \tag{2.36}$$

Notice that the renormalizing subtraction $q^R \equiv q_F$ has been determined without choosing any vacuum state. Since q_F is a numerical quantity, independent of ϕ_1 and ϕ_2, the subtraction is a c number which does not change (2.23). But the nonlinear commutator (2.24) is modified as in (2.14), and the Lie algebra of the renormalized generators acquires a central extension:

$$[:Q_f:\ Q_g:] = i:Q_{(f,g)} + \frac{c}{24\pi} \int f' k g'$$

$$= i:Q_{(f,g)}: - \frac{i}{48\pi} c \int (fg''' - gf''') , \quad c = 1 \ . \tag{2.37}$$

Of course, the subtraction is ambiguous up to terms that are finite in the local limit: these are obviously "trivial" in the sense that they may be adjusted at will by a finite redefinition of the generators. But the result for the nontrivial part of the extension—not removable by redefining generators—is unique and it is not specific to the representation (2.27) for χ, which may be generalized to

$$\langle \phi_1 | X | \phi_2 \rangle = -\frac{i}{\sqrt{2}} \left[\alpha \frac{\delta}{\delta \phi_1} + \beta k \phi_1 \right] \delta(\phi_1 - \phi_2) \tag{2.38a}$$

provided

$$\tfrac{1}{2}(\alpha k \beta^T + \beta k \alpha^T) = k \tag{2.38b}$$

as required by (2.21). One may verify that the representation kernel which arises from the more general formula (2.38) possesses a different Gaussian ϕ_1, ϕ_2 dependence and the infinite constant Z is modified. But the F-dependent infinity (2.33) is unaffected by the generalization (2.38), so the center in (2.37) remains the same.[6] Finally, note that in our derivation the sign of the center would change if the analysis of divergences were performed after an unconventional continuation to imaginary τ: $\tau F \to i\tau F$ rather than $\tau F \to -i\tau F$. The latter choice, which we made, is appropriate to theories with energy spectra bounded below, but not above.

2. Conventional method

The above approach is to be contrasted with the conventional one, wherein the subtraction is given by the expectation of Q_F in a Fock vacuum of the form (2.3). (For simplicity, here we take Ω to be real.) The problem is that without specifying a dynamical Hamiltonian, which determines a unique ground state, the covariance is undetermined. The expectation value of Q_F in the state $|\Omega\rangle$ is

$$\begin{aligned} q_F^\Omega &\equiv \langle \Omega | Q_F | \Omega \rangle \\ &= \tfrac{1}{4} \operatorname{tr} F(\Omega - (\Omega - k)\rho(\Omega + k)) , \end{aligned} \tag{2.39}$$

where ρ is the two-point function for the field:

$$\rho(x,y) \equiv \langle \Omega | \phi(x)\phi(y) | \Omega \rangle = \tfrac{1}{2}\Omega^{-1}(x,y) . \tag{2.40}$$

The conventional subtraction therefore depends on Ω,

$$q_F^\Omega = \tfrac{1}{8} \operatorname{tr} F(\Omega + k\Omega^{-1}k) , \tag{2.41}$$

as do the conventionally renormalized generators:

$$:Q_F^\Omega: \equiv \lim_{F \to f} (Q_F - q_F^\Omega) . \tag{2.42}$$

The Ω dependence survives in the center of the algebra (2.24). For example, for translation-invariant vacua,

$$\Omega(x,y) = \int \frac{dp}{2\pi} e^{-ip(x-y)} \Omega(p) , \tag{2.43}$$

the last term in (2.37) is replaced by

$$-\frac{i}{48\pi} \int (fg''' - gf''') \to \tfrac{1}{4} \int dx\, dy\, f(x) g(y) \int \frac{dp}{2\pi} e^{-ip(x-y)} \int \frac{dq}{2\pi} \left[\left[q + \frac{p}{2} \right] C \left[q - \frac{p}{2} \right] - \left[q - \frac{p}{2} \right] C \left[q + \frac{p}{2} \right] \right] , \tag{2.44}$$

where C is constructed from Ω:

$$C(p) \equiv \frac{|p|}{2} \left[\frac{\Omega(p)}{|p|} + \frac{|p|}{\Omega(p)} \right] . \tag{2.45}$$

With our approach, a unique (up to finite terms) covariance is selected: $\Omega = \omega$, $C(p) = |p|$.

3. Discussion

We have seen that the intrinsic renormalization in field space produces a unique result for the center, in contrast with the state-dependent answer which emerges with the vacuum subtraction. The reason for the more specific result is that in the intrinsic approach we have constructed a representation for the *finite* transformation group, rather than only for the *infinitesimal* Lie algebra; i.e., our generators can be exponentiated.

In the conventional vacuum subtraction method for renormalizing the conformal algebra, one may place additional regularity requirements, which limit the allowed vacua—the allowed covariances Ω—and within this limited class, the center is essentially independent of Ω. This further program cannot be carried out in all cases. For example, in de Sitter space, only our intrinsic method is available.[4] However, for the specific problem of the conformal algebra in Minkowski space, the additional development is interesting and we now discuss it. We restrict ourselves to translation invariant Ω, so that (2.44) and (2.45) are relevant, and we assume that no singularities arise at finite momenta.

First, it is natural to demand that the center be finite; i.e., the q integral in (2.44) converges. Note that (2.45) implies that $C(p)$ cannot grow more slowly than $|p|$ for large p; therefore each of the two terms in the q integration of (2.44) produces at least a cubic divergence. Consequently, the integration variable must not be separately shifted in the two terms. Indeed if separate shifts were made, cancellations could be effected and one would conclude that the center is proportional to $\int (fg' - gf')$. This could be removed by redefining the generators—it is trivial. Growth with p faster than $|p|$ renders the center infinite. So if we require the best possible behavior, Ω is restricted to behave as

$$\Omega(p) \underset{p \to \infty}{\propto} |p| \tag{2.46a}$$

and

$$C(p) \underset{p\to\infty}{\propto} |p| \; . \tag{2.46b}$$

[If $\Omega(p)=n\,|p\,|$, the center is as in (2.37) except that $c=\frac{1}{2}(n+1/n)\geq 1$ (Ref. 5).]

Second, we consider the variance of the generator:

$$(\Delta Q_f)^2=\langle\Omega\,|\,(Q_f^2-\langle\Omega\,|\,Q_f\,|\,\Omega\rangle^2)\,|\,\Omega\rangle \; . \tag{2.47}$$

This is also recognized as the norm of the state $:Q_f:|\,\Omega\rangle$. One finds

$$(\Delta Q_f)^2=\frac{1}{8}\int dx\,dy f(x)f(y)\int \frac{dp}{2\pi}e^{-ip(x-y)}\int\frac{dq}{2\pi}[C(p+q)C(q)-(p+q)q] \; . \tag{2.48}$$

It is natural to demand that this quantity be finite, which requires C and Ω to behave asymptotically as $|\,p\,|$ (i.e., n above must be unity).

Thus the two above requirements, the second being necessary for the operator $:Q_f:$ to be self-adjoint in the Fock space constructed on a vacuum with covariance Ω, fix the asymptotic behavior of the covariance. Clearly, covariances that differ only in the nonasymptotic region give rise to equivalent Fock spaces. Moreover, they produce centers that differ by trivial terms. This is so because if Ω_1 and Ω_2 lead to C_1 and C_2 and C_1-C_2 decreases rapidly, integration shifts may be performed in the formula for the difference of two centers [Eq. (2.44) with C replaced by C_1-C_2]. The result then is trivial as explained above.

In conclusion, note that our functional transformation kernel allows computing how states transform under conformal transformations:

$$\Psi(\phi)\underset{F}{\rightarrow}\Psi_F(\phi)=e^{iq_F}\int \mathcal{D}\bar{\phi}\,U(\phi,\bar{\phi};F)\Psi(\bar{\phi}) \; . \tag{2.49}$$

In particular, for a Gaussian with covariance Ω, the transformed state is again Gaussian with transformed covariance:

$$\Omega_F=\Omega-(\Omega+k)(\Omega-iK_F)^{-1}(\Omega-k) \; . \tag{2.50a}$$

Also the transformed state acquires an additional phase θ_F:

$$\theta_F=q_F+\mathrm{Im}[\ln N_F-\tfrac{1}{2}\,\mathrm{tr}\,\ln(\Omega-iK_F)] \; . \tag{2.50b}$$

The Fock vacuum of a massless theory possesses $\Omega=\omega$, which is invariant for the SO(2,1) subgroup of conformal transformations generated by $f(x)=(1,x,x^2)$ (Ref. 1). More generally, Ω is a representation for the conformal algebra without center; the latter resides in the representation provided by the phase θ_F (Ref. 1).

III. FERMIONIC FIELD THEORIES

A development for fermion theories, analogous to the one in Sec. II for boson theories, is presented here. We shall discuss charge-neutral Majorana fermions, $\psi=\psi^\dagger$, while charged fermions of the Weyl (massless) or Dirac (massive) variety can be described by a pair of neutral ones: $\psi=(1/\sqrt{2})(\psi_1+i\psi_2)$, $\psi^\dagger=(1/\sqrt{2})(\psi_1-i\psi_2)$. We shall be concerned with theories in two space-time dimensions, i.e., fermions on a line, where a Majorana or Weyl field has only one component: $\psi=\psi(x)$. Our ideas may be applied in other dimensions, but regularization results may differ owing to differences in singularities.[7] Two dimensions are simplest; moreover these days it is in vogue because of the string. For an illustration in an even simpler setting, we shall also discuss fermions on a space with a finite number of points: $\psi=\psi(i)$; this is just fermionic quantum mechanics.

A. The function space

Our function space consists of functionals $\Psi(u)$ of a Grassmann field $u(x)$ at fixed time,

$$\{u(x),u(y)\}=0 \; , \tag{3.1}$$

and we associate each functional with the ket $|\,\Psi\rangle\leftrightarrow\Psi(u)$. We need a rule for realizing on this space the operator $\psi(x)$, which is Hermitian and satisfies canonical anticommutation relations:

$$\psi^\dagger=\psi \; , \tag{3.2}$$

$$\{\psi(x),\psi(y)\}=\delta(x-y)\equiv I(x,y) \; . \tag{3.3}$$

In other words, we seek a representation of the (infinite) Clifford algebra (3.3) in terms of Grassmann variables (3.1). To this end, we represent the action of $\psi(x)$ on the state $|\,\Psi\rangle$ by

$$\psi(x)\,|\,\Psi\rangle\leftrightarrow\frac{1}{\sqrt{2}}\left[u(x)+\frac{\delta}{\delta u(x)}\right]\Psi(u) \tag{3.4}$$

and thereby satisfy (3.3). To verify (3.2), we must define an inner product on the functional space with respect to which the operator in (3.4) is Hermitian.

The inner product involves a Grassmann integration over u of an element in the functional space, composed with an element in the dual functional space. For the bosonic case, the dual is constructed by complex conjugation. Here this will not suffice, as the following example shows. It is possible for a functional to be u independent. If the dual functional were its complex conjugate, the inner product would vanish, since a Grassmann u integral over a u-independent quantity is zero. The "state" would have zero norm, and this is undesirable.

To understand how the dual must be constructed, we analyze first the problem on a space $\{x\}$ consisting of two points with two fermion operators $\psi(i)$, $i=1,2$, satisfying a Clifford algebra:

$$\{\psi(i),\psi(j)\}=\delta_{ij}\ . \tag{3.5}$$

(When we model generic continuum field theories with a discrete example, we use an even number of discrete points. This is appropriate to the charge-conjugation-invariant situation with no unpaired, charge self-conjugate states.) A specific state $|\Psi_f\rangle$ is represented by a function of $u(i)$ that can be expanded in a four-dimensional basis:

$$\begin{aligned}|\Psi_f\rangle\leftrightarrow\Psi_f(u)&=f_0+\sum_i f_1(i)u(i)\\&\quad+\frac{1}{2}\sum_{i,j}f_2(i,j)u(i)u(j)\\&=f_0+f_1(1)u(1)+f_1(2)u(2)\\&\quad+f_2(1,2)u(1)u(2)\ .\end{aligned} \tag{3.6}$$

The f_i's are numbers with $f_2(1,2)=-f_2(2,1)$. The inner product with a second state $|\Psi_g\rangle$ is defined in the natural way:

$$\begin{aligned}\langle\Psi_g|\Psi_f\rangle&=g_0^*f_0+\sum_i g_1^*(i)f_1(i)\\&\quad+\tfrac{1}{2}\sum_{i,j}g_2^*(i,j)f_2(i,j)\\&=\langle\Psi_f|\Psi_g\rangle^*\ .\end{aligned} \tag{3.7}$$

This can be expressed as

$$\langle\Psi_g|\Psi_f\rangle=\int d^2u\ \Psi_g^*(u)\Psi_f(u) \tag{3.8}$$

provided the dual of $|\Psi_g\rangle$ is represented by[8]

$$\begin{aligned}\langle\Psi_g|\leftrightarrow\Psi_g^*(u)&=g_2^*(1,2)+g_1^*(2)u(1)\\&\quad-g_1^*(1)u(2)+g_0^*u(1)u(2)\ .\end{aligned} \tag{3.9}$$

Equation (3.7) follows from (3.6), (3.8), and (3.9) since only one Grassmann integral is nonvanishing:

$$\int d^2u\ u(1)u(2)=1\ . \tag{3.10}$$

Thus the star operation signifies complex conjugation on numbers such as f_n and g_n, but it dualizes Grassmann variables in a way similar to differential forms.[9] Since the two-dimensional Grassmann δ functions is also given by a product,

$$\delta^2(u-\bar u)=[u(1)-\bar u(1)][u(2)-\bar u(2)]\ , \tag{3.11a}$$

$$\int d^2\bar u\ \delta^2(u-\bar u)\Psi(\bar u)=\Psi(u)\ , \tag{3.11b}$$

the dual (3.9) may be written as

$$\Psi_g^*(u)=\left[g_0^*-\sum_i g_1^*(i)\frac{\delta}{\delta u(i)}+\tfrac{1}{2}\sum_{i,j}g_2^*(i,j)\frac{\delta^2}{\delta u(j)\delta u(i)}\right]\delta^2(u)\ . \tag{3.12a}$$

Another way of representing our dual state is the following. Introduce auxiliary variables $\bar u(i)$; also define an "intermediate" dual functional $\bar\Psi_g(\bar u)$:

$$\begin{aligned}\bar\Psi_g(\bar u)&=g_0^*+g_1^*(1)\bar u(1)+g_1^*(2)\bar u(2)\\&\quad+g_2^*(1,2)\bar u(2)\bar u(1)\ .\end{aligned} \tag{3.12b}$$

Then our $\Psi_g^*(u)$ in (3.9) or (3.12a) is given by the Berezin integral

$$\Psi_g^*(u)=\int d\bar u(2)d\bar u(1)\exp\left[\sum_i\bar u(i)u(i)\right]\bar\Psi_g(\bar u)\ . \tag{3.12c}$$

With these definitions, the Hermitian conjugate of $u(i)$ is $\delta/\delta u(i)$ and ψ, given by

$$\psi(i)=\frac{1}{\sqrt2}\left[u(i)+\frac{\delta}{\delta u(i)}\right], \tag{3.13}$$

is Hermitian.

Of particular interest are the states $|\Omega\rangle$ which are represented by Gaussian functions:

$$\begin{aligned}|\Omega\rangle\leftrightarrow\Psi_\Omega(u)&=\det{}^{-1/4}\Omega\exp[\tfrac{1}{2}(u\Omega u)]\\&=\frac{1}{\sqrt{\Omega_{12}}}[1+\Omega_{12}u(1)u(2)]\ .\end{aligned} \tag{3.14}$$

(Here Ω is an antisymmetric 2×2 matrix, hence it possesses one entry Ω_{12}.) Using (3.9) or (3.12) one finds that the dual state is represented by

$$\begin{aligned}\langle\Omega|\leftrightarrow\Psi_\Omega^*(u)&=\det{}^{-1/4}(\Omega^{\dagger-1})\exp[\tfrac{1}{2}(u\Omega^{\dagger-1}u)]\\&=\frac{1}{\sqrt{\Omega_{12}^*}}[u(1)u(2)+\Omega_{12}^*]\ .\end{aligned} \tag{3.15}$$

Note that these states are not normalized to unity:

$$\langle\Omega|\Omega\rangle=\det{}^{1/2}(\Omega^{1/2}\Omega^{\dagger1/2}+\Omega^{-1/2}\Omega^{\dagger-1/2})\ . \tag{3.16}$$

Also of interest are polynomials in u multiplying $|\Omega\rangle$. These can be generated from a Gaussian with a Grassmann source:

$$|\Omega;f\rangle \equiv \left[f_0+\sum_i f_1(i)u(i)+\tfrac{1}{2}\sum_{i,j} f_2(i,j)u(i)u(j)\right]|\Omega\rangle$$
$$\leftrightarrow \left[f_0+\sum_i f_1(i)\frac{\delta}{\delta J(i)}+\tfrac{1}{2}\sum_{i,j} f_2(i,j)\frac{\delta}{\delta J(i)\delta J(j)}\right]\det{}^{-1/4}\Omega\exp[\tfrac{1}{2}(u\Omega u)+(Ju)]\Big|_{J=0}. \tag{3.17}$$

The dual is given by

$$\langle\Omega,f|\leftrightarrow\left[f_0^*+\sum_i f_1^*(i)\frac{\delta}{\delta J(i)}+\tfrac{1}{2}\sum_{i,j} f_2^*(i,j)\frac{\delta^2}{\delta J(j)\delta J(i)}\right]\det{}^{-1/4}(\Omega^{\dagger-1})\exp[\tfrac{1}{2}(u-J)\Omega^{\dagger-1}(u-J)]\Big|_{J=0}. \tag{3.18}$$

Further properties of this formalism are discussed in the Appendix.

The representation (3.13) of the Clifford algebra (3.5) is four dimensional, as is seen from (3.6). Consequently, it is reducible, since a two-dimensional, irreducible representation is given by any two Pauli matrices: $\psi(i)=\sigma^i/\sqrt{2}$. More generally, a $2n$-dimensional Clifford algebra possess an irreducible matrix representation with dimensionality 2^n, while our formalism gives a reducible, 2^{2n}-dimensional representation.

It is possible to give an irreducible representation in terms of Grassmann variables by splitting the Clifford elements in two, representing half of them as in (3.13) and the other half by $(1/\sqrt{2}i)[u(i)-\delta/\delta u(i)]$, which is also Hermitian, satisfies (3.5) and anticommutes with (3.13). However, for the continuum field theory we do not adopt this approach for the following two reasons. First, there is no *a priori* natural choice for the splitting. [When a Hamiltonian is posited, one could effect a splitting by reference to the positive and negative frequencies—this is essentially the holomorphic representation (see the Appendix). But our whole purpose is to develop representation theory without reference to dynamics. Moreover, the division into positive and negative frequencies can change if the Hamiltonian is time dependent, or depends on other varying parameters.] Second, as will be discussed below, the reducibility seems desirable, since it allows making inequivalent choices for filling the Dirac sea when defining the vacuum of a dynamical model.

For the continuum field theory we use the above results extended to a continuous infinite of points. A member of the functional space

$$|\Psi_f\rangle\leftrightarrow\Psi_f(u)$$
$$=f_0+\int f_1(x)u(x)$$
$$+\tfrac{1}{2}\int f_2(x_1,x_2)u(x_1)u(x_2)+\cdots \tag{3.19}$$

possesses the dual

$$\langle\Psi_f|\leftrightarrow\Psi_f^*(u)$$
$$=\left[f_0^*-\int f_1^*(x)\frac{\delta}{\delta u(x)}\right.$$
$$\left.+\tfrac{1}{2}\int f_2^*(x_1,x_2)\frac{\delta^2}{\delta u(x_2)\delta u(x_1)}+\cdots\right]\delta(u), \tag{3.20}$$

where the functions f_n are totally antisymmetric in their arguments. The inner product is defined by functional Grassmann integration:

$$\langle\Psi_g|\Psi_f\rangle=\int\mathcal{D}u\,\Psi_g^*(u)\Psi_f(u)$$
$$=\langle\Psi_f|\Psi_g\rangle^*. \tag{3.21}$$

For a Gaussian state, which we also call a Fock vacuum in analogy with the bosonic case,

$$|\Omega\rangle\leftrightarrow\Psi_\Omega(u)=\det{}^{-1/4}\Omega\exp\left[\tfrac{1}{2}\int u\Omega u\right], \tag{3.22}$$

the dual is

$$\langle\Omega|\leftrightarrow\Psi_\Omega^*(u)=\det{}^{-1/4}(\Omega^{\dagger-1})\exp\left[\tfrac{1}{2}\int u\Omega^{\dagger-1}u\right]. \tag{3.23}$$

Here Ω is an antisymmetric kernel. A polynomial in u, multiplying the Gaussian

$$|\Omega,f\rangle=\left[f_0+\int f_1(x)u(x)\right.$$
$$\left.+\tfrac{1}{2}\int f_2(x_1,x_2)u(x_1)u(x_2)+\cdots\right]|\Omega\rangle \tag{3.24}$$

can be generated with a Gaussian source,

$$|\Omega,f\rangle\leftrightarrow\left[f_0+\int f_1(x)\frac{\delta}{\delta J(x)}+\frac{1}{2}\int f_2(x_1,x_2)\frac{\delta}{\delta J(x_1)\delta J(x_2)}+\cdots\right]\det{}^{-1/4}\Omega\exp\left[\int(\tfrac{1}{2}u\Omega u+Ju)\right]\Big|_{J=0}, \tag{3.25}$$

while its dual is generated by

$$\langle \Omega, f| \leftrightarrow \left[f_0^* + \int f_1^*(x) \frac{\delta}{\delta J(x)} + \frac{1}{2} \int f_2^*(x_1,x_2) \frac{\delta}{\delta J(x_2)\delta J(x_1)} + \cdots \right]$$

$$\times \det{}^{-1/4}(\Omega^{\dagger -1}) \exp\left[\tfrac{1}{2} \int (u-J)\Omega^{\dagger -1}(u-J) \right] \Big|_{J=0} . \tag{3.26}$$

Of course the field-theoretic dual may also be represented by a Berezin integral, as in (3.12b) and (3.12c).

The above concerns charge-neutral Majorana fermions. Charged fermions $(\psi,\psi^\dagger)$ are described by a pair of Majorana fields (ψ_1,ψ_2),

$$\psi = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2), \quad \psi^\dagger = \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2) , \tag{3.27}$$

so their representation is constructed accordingly: functionals depend on u_1 and u_2 or equivalently on $u=(1/\sqrt{2})(u_1+iu_2)$ and $u^\dagger=(1/\sqrt{2})(u_1-iu_2)$, while operators are realized by

$$\psi = \frac{1}{\sqrt{2}}\left[u + \frac{\delta}{\delta u^\dagger} \right], \quad \psi^\dagger = \frac{1}{\sqrt{2}}\left[u^\dagger - \frac{\delta}{\delta u} \right] , \tag{3.28}$$

etc.

Because our inner product is numerically valued, there is no field basis for our functional space: functionals of u are not overlaps with field states and operator kernels are not matrix elements. This is unlike the bosonic case.

B. Two-dimensional conformal transformations

Two-dimensional conformal transformations of the type discussed in Sec. II C above, act also on fermion Majorana fields and can be represented by these variables. The formal generator

$$Q_f = \frac{i}{4} \int dx [\psi(x) f(x) \psi'(x) - \psi'(x) f(x) \psi(x)] \tag{3.29}$$

gives the field transformation law

$$\delta_f \psi = i[Q_f,\psi] = (f\psi)' - \tfrac{1}{2} f' \psi . \tag{3.30}$$

Q_f formally satisfies the algebra (2.24), but suffers from singularities owing to the coincident-point operator product. For a well-defined regularized generator we take

$$Q_F = \tfrac{1}{2} \int \psi F \psi . \tag{3.31}$$

Equation (3.29) is regained when the antisymmetric Hermitian kernel $F(x,y)$ tends to

$$F(x,y) \to \frac{i}{2}[f(x)+f(y)]\delta'(x-y)$$

$$= \tfrac{1}{2}[f(x)+f(y)]k(x,y) . \tag{3.32}$$

[Notice that this is as (2.26) with the interchange $k \leftrightarrow \delta$; see also below.] Q_F satisfies

$$[Q_F,Q_G] = iQ_{(F,G)}, \quad (F,G) = -i[F,G] . \tag{3.33}$$

The representation and intrinsic renormalization of these quantities is an important application of our formalism.

The kernel $U(u_1,u_2;F)$ that represents the finite transformation satisfies the differential equation

$$i\frac{\partial}{\partial \tau} U(u_1,u_2;\tau F) = \tfrac{1}{4} \int dx\, dy \left[u_1(x) + \frac{\delta}{\delta u_1(x)} \right] \times F(x,y) \left[u_1(y) + \frac{\delta}{\delta u_1(y)} \right] \times U(u_1,u_2;\tau F) \tag{3.34}$$

with a boundary condition at $\tau=0$:

$$U(u_1,u_2;0) = \delta(u_1-u_2) . \tag{3.35}$$

The solution is Gaussian:

$$U(u_1,u_2;F) = N_F \exp\left[-\int u_1 u_2 \right] \times \exp\left[\frac{i}{2} \int (u_1-u_2) K_F (u_1-u_2) \right] , \tag{3.36}$$

$$N_F = \det{}^{1/2} i \sin\frac{F}{2} , \tag{3.37}$$

$$K_F = \cot\frac{F}{2} . \tag{3.38}$$

Constants are adjusted in N_F so that (3.35) holds. That U is indeed the correct transformation kernel, satisfying the composition law

$$\int \mathcal{D}u\, U(u_1,u;F) U(u,u_2;G) = U(u_1,u_2;F\circ G) , \tag{3.39}$$

$$F \circ G = F + G + \tfrac{1}{2}(F,G) + \cdots , \tag{3.40}$$

can be verified explicitly when F is proportional to G; the more general case has been checked up to third order in the expansion of $F\circ G$.

The definition of the inner product on our space determines the form of the adjoint kernel:

$$U^{\dagger}(u_1,u_2;F)=N_F^* \exp\left[-\int u_1 u_2\right]\exp\left[-\frac{i}{2}\int (u_1-u_2)K_F^{\dagger}(u_1-u_2)\right] . \tag{3.41}$$

Since F and K_F are Hermitian, the above is just $U(u_1,u_2;-F)$; hence, the representation is unitary.

The kernel (3.36) should be compared to the corresponding bosonic one (2.28). The first exponential is similar with the commutator of the bosonic fields, k, replaced by the anticommutator of the fermionic fields, $I\equiv\delta$. The analogy of the remaining formulas is brought out, if we similarly replace k in $\mathcal{F}$ by I, i.e., replace $\mathcal{F}$ by F. Of course, differences in the Jacobian factor between fermions and bosons have to be taken into account.

The local limit when the regulator is removed can be as evaluated as in the bosonic case. K_F attains a well-defined expression

$$\begin{aligned} K_F(x,y)\to K_f(x,y)&=\frac{1}{\sqrt{f(x)}}\left[\int\frac{d\lambda}{2\pi}\left[\cot\frac{\lambda}{2}\right]\exp\left[-i\lambda\int_y^x\frac{dz}{f(z)}\right]\right]\frac{1}{\sqrt{f(y)}}\\ &=\frac{1}{\sqrt{f(x)}}\left[P\cot\pi\int_y^x\frac{dz}{f(z)}\right]\frac{1}{\sqrt{f(y)}} .\end{aligned} \tag{3.42}$$

The normalization factor diverges. The analysis, as in the bosonic case, is performed for imaginary $\tau(\tau F\to -i\tau F)$, where N_F becomes $e^{[\mathrm{tr}\ln\sinh(|F|/2)]/2}$ and the antisymmetric kernel F is replaced by the absolute value kernel. The result, after returning to real τ, is

$$N_F\to Z\exp\left[i\frac{\pi}{12}\int\frac{dx}{f(x)}\right]e^{-iq_F} ,$$

$$Z=\det{}^{-1/2}2 , \tag{3.43}$$

$$\begin{aligned} q_F&=-\tfrac{1}{4}\mathrm{tr}\,|F|\\ &=-\tfrac{1}{4}\mathrm{tr}F\frac{k}{|k|}+(\text{terms that vanish in the local limit}) . \end{aligned}\tag{3.44}$$

The last equality is obtained by manipulations similar to those of the bosonic case.

To renormalize, we absorb the constant divergence Z in the definition of functional integration measure, and remove the divergent phase. Thus $Z^{-1}e^{iq_F}U(u_1,u_2;F)$ possesses a finite limit, but the composition law (3.39) acquires a trivial cocycle,

$$\omega_2(F,G)=\tfrac{1}{4}\mathrm{tr}(F\circ G-F-G)\frac{k}{|k|} \tag{3.45}$$

which becomes nontrivial in the local limit, where its infinitesimal form is

$$\delta\omega_2(f,g)=\lim\left[-\frac{i}{8}\mathrm{tr}[F,G]\frac{k}{|k|}\right] . \tag{3.46}$$

This implies that the renormalized charges

$$:Q_f:=\lim\left[Q_F+\tfrac{1}{4}\mathrm{tr}F\frac{k}{|k|}\right] \tag{3.47}$$

satisfy (2.37) with $c=\frac{1}{2}$, in agreement with a general theorem.[10]

As in the bosonic case this central extension is not sensitive to the way that the field operator is represented. One can verify that the divergent phase in (3.43) is unaffected by the following generalization of (3.4):

$$\psi=\frac{1}{\sqrt{2}}\left[\alpha u+\alpha^*\frac{\delta}{\delta u}\right] , \tag{3.48a}$$

$$\tfrac{1}{2}(\alpha^*\alpha^T+\alpha\alpha^{\dagger})=I , \tag{3.48b}$$

the last condition being required by (3.3). But also, as in the bosonic case, the opposite sign for the center results if the analysis of divergences is carried out after continuation to imaginary τ in the opposite sense from ours: $\tau F\to i\tau F$ rather than $\tau F\to i\tau F$, the latter being the conventional one for theories with energy spectra bounded below but not above.

The action of the transformation kernel on a generic state $\Psi(u)$ is given by

$$\Psi(u)\to\Psi_F(u)=e^{iq_F}\int\mathcal{D}\tilde{u}\;U(u,\tilde{u};F)\Psi(\tilde{u}) . \tag{3.49}$$

In particular, the transform of the Gaussian (3.22), properly normalized, is again a Gaussian with transformed covariance,

$$\Omega_F=\Omega+(I-\Omega)(\Omega+iK_F)^{-1}(I+\Omega) , \tag{3.50a}$$

and an additional phase θ_F:

$$\theta_F=q_F+\mathrm{Im}[\ln N_F+\tfrac{1}{2}\mathrm{tr}\ln(\Omega+iK_F)] . \tag{3.50b}$$

As in the boson case [compare (2.50)], Ω is a representation for the conformal algebra without center; the latter resides in the representation provided by the phase θ_F.

Charged fermions give similar results. The formal generator is

$$Q_f=\frac{i}{2}\int dx[\psi^{\dagger}(x)f(x)\psi'(x)-\psi'^{\dagger}(x)f(x)\psi(x)] \tag{3.51}$$

and the transformation kernel is modified from (3.36) in an obvious way, due to the doubling in degrees of freedom:

$$U(u_1^\dagger,u_1,u_2^\dagger,u_2;F)=N_F^2\exp\left[-\int(u_1^\dagger u_2-u_2^\dagger u_1)\right]\exp\left[i\int(u_1^\dagger-u_2^\dagger)K_F(u_1-u_2)\right]. \tag{3.52}$$

As a consequence the center on the algebra is twice the Majorana value, i.e., $c=1$ in (2.37) (see Ref. 11).

C. Fock-space dynamics

In the previous subsection our discussion of the two-dimensional conformal group exemplified representing transformation groups on fermions without reference to any specific dynamical Hamiltonian. Now we examine dynamical second-quantized fermion theories in our formalism. This we do in order to explore further properties of our fermionic Schrödinger picture and also to give explicit realization to some of the peculiarities of fermion field theory: Dirac sea, fractional charge, anomalies, and Berry's phase.

Let us begin by recalling the algebraic structures associated with the field-theoretic Majorana-Weyl Hamiltonian:

$$H=\tfrac{1}{2}\int\psi h\psi\ . \tag{3.53}$$

The "first quantized" Hamiltonian h is antisymmetric and imaginary,

$$h(x,y)=-h^*(x,y)\ , \tag{3.54}$$

with a complete, orthonormal set of "first quantized" eigenmodes:

$$hf_\lambda=\lambda f_\lambda\ , \tag{3.55}$$

$$\int dx\, f_{\lambda'}^*(x)f_\lambda(x)=\delta(\lambda-\lambda')\ , \tag{3.56a}$$

$$\int d\lambda\, f_\lambda^*(x)f_\lambda(y)=\delta(x-y)\ . \tag{3.56b}$$

(Whenever appropriate, summation over discrete eigenvalues and replacement of the δ function by a Kronecker delta is understood.) According to (3.54), the eigenvalues are paired in sign,

$$hf_\lambda^*=-\lambda f_\lambda^*\ , \tag{3.57}$$

and we assume that there are no isolated vanishing eigenvalues. The field operator may be expanded in the first-quantized modes:

$$\psi(x)=\int d\lambda\, a_\lambda f_\lambda(x)\ , \tag{3.58}$$

$$a_\lambda=a_{-\lambda}^\dagger=\int dx\, f_\lambda^*(x)\psi(x)\ , \tag{3.59}$$

$$\{a_\lambda,a_{-\lambda'}\}=\{a_\lambda,a_{\lambda'}^\dagger\}=\delta(\lambda-\lambda')\ . \tag{3.60}$$

The mode operator a_λ is a shift operator for the second-quantized Hamiltonian:

$$[a_\lambda,H]=\lambda a_\lambda\ . \tag{3.61}$$

Hence, the spectrum of H is unbounded from above and below, unless a_λ annihilates states. Of course, all this is familiar, and is realized, for example, for the Poincaré-invariant model:

$$h(x,y)=i\delta'(x-y)\ . \tag{3.62}$$

We now seek eigenstates of H within our Grassmann functional space. For the Fock vacuum we choose a Gaussian,

$$|\Omega\rangle=\det{}^{-1/4}\Omega\exp\left[\tfrac{1}{2}\int u\Omega u\right], \tag{3.63}$$

where Ω is antisymmetric. The eigenvalue equation

$$H|\Omega\rangle=\tfrac{1}{4}\int\left[u+\frac{\delta}{\delta u}\right]h\left[u+\frac{\delta}{\delta u}\right]|\Omega\rangle=E_V|\Omega\rangle \tag{3.64}$$

requires that

$$(I-\Omega)h(I+\Omega)=0 \tag{3.65}$$

and the vacuum energy is

$$E_V=\tfrac{1}{4}\operatorname{tr}h\Omega\ . \tag{3.66}$$

Excited states are polynomials in ψ operating on $|\Omega\rangle$, which in our formalism become polynomials in $\frac{1}{2}(I+\Omega)u\equiv u_+$ multiplying the Fock vacuum.

We now show that Ω is not determined uniquely by (3.65). Take Ω to be simultaneously diagonalized with h, so (3.65) requires $\Omega^2=I$; in the λ representation, h is diagonal and we have

$$\Omega(\lambda,\lambda')=\Omega(\lambda)\delta(\lambda-\lambda')\ , \tag{3.67a}$$

$$\Omega(\lambda)=-\Omega(-\lambda)\ , \tag{3.67b}$$

$$\Omega(\lambda)=\pm1\ , \tag{3.67c}$$

where the variation in sign can occur for any matrix element. In other words, there is an infinity of solutions for Ω depending on the different ways one assigns signature.

To understand further the form of Ω, and to select one from the infinity, we compute the effect of a_λ:

$$\begin{aligned}a_\lambda|\Omega\rangle&=\int f_\lambda^*\psi|\Omega\rangle\\&=\frac{1}{\sqrt2}\int f_\lambda^*\left[u+\frac{\delta}{\delta u}\right]|\Omega\rangle\\&=\frac{1}{\sqrt2}[1+\Omega(\lambda)]\int f_\lambda^* u|\Omega\rangle\ .\end{aligned} \tag{3.68}$$

From (3.68) it is seen that a_λ annihilates $|\Omega\rangle$ whenever $\Omega(\lambda)$ is -1. Thus choosing Ω is equivalent to choosing the prescription for filling the Dirac sea to define a field-theoretic vacuum. When $\Omega(\lambda)=-1$ for positive λ and $+1$ for negative λ, i.e.,

$$\Omega(\lambda)=\epsilon(-\lambda)\ , \tag{3.69}$$

a_λ annihilates $|\Omega\rangle$ for $\lambda>0$ but not for $\lambda<0$. This is the conventional choice of filled negative-energy sea, and corresponds to

$$E_V = -\frac{V}{2}\int_0^\infty d\lambda\,\lambda\ , \tag{3.70}$$

where V is the volume of space.

More generally, $\frac{1}{2}[1+\Omega(\lambda)]$ is the filling factor, vanishing for empty states. Choices other than (3.69) for Ω are also possible, corresponding to other filling prescriptions, but they would be unconventional, and would in general define inequivalent theories. Note that the overlap between two vacua is proportional to

$$\langle \Omega_1 \mid \Omega_2\rangle \propto \det{}^{1/2}(\Omega_1+\Omega_2)\ . \tag{3.71}$$

Since Ω_1 and Ω_2 can differ only in the sign of one or more eigenvalues, $\Omega_1+\Omega_2$ has a zero eigenvalue. Whether the overlap vanishes, depends on the weight of the zero in $\Omega_1+\Omega_2$. If different signs are associated to a discrete mode in Ω_1 and Ω_2, then the determinant certainly vanishes; for continuum modes, a sufficiently infinite number of modes must be differently filled between Ω_1 and Ω_2 for the determinant to vanish. When the overlap of two vacua vanishes, so will the overlap between corresponding excited states, and the Fock spaces—the different theories built with sufficiently different Ω's—are inequivalent.

Let us further observe that for the energy of states built by multiplication of $|\Omega\rangle$ by ψ will in general depend on Ω. But repeated application of the shift operator a_λ will result in the same energy spectrum being attained regardless of the choice of Ω. Hence, there is a large degeneracy in our formalism—this comes from the reducibility of our representation. However, we see no difficulty with this (see below); on the contrary, the degeneracy reflects the true circumstances that in a fermionic quantum field theory, with a given Hamiltonian, a prescription for defining a vacuum must also be chosen.

Therefore, in our Schrödinger picture the first step for defining a theory, even with specific dynamics in hand, is choosing Ω. But there remains yet another subtlety, another degeneracy. As mentioned before, higher Fock states are polynomials in $\frac{1}{2}(I+\Omega)u\equiv u_+$ multiplying the vacuum Gaussian. However, we may also consider polynomials in $\frac{1}{2}(I-\Omega)u\equiv u_-$. Because $\frac{1}{2}(I\pm\Omega)$ are projection operators, polynomials in u_+ are orthogonal to those with u_-. Also multiplying by u_- does not affect the energy of a state. But there is no operator constructed from ψ which can produce factors of u_-. [The projection u_- arises when $|\Omega\rangle$ is operated on by $(1/\sqrt{2}i)(u-\delta/\delta u)$; but this combination does not correspond to any operator constructed from $\psi\sim(1/\sqrt{2})(u+\delta/\delta u)$.] Hence, we may safely ignore states involving polynomials in u_-, provided we remember to consider only operations with the Fermi fields. (Note, however, that the Gaussian *does* contain u_-, since $\frac{1}{2}u\,\Omega u=u_-u_+$; consequently, by varying Ω one *does* produce polynomials in u_-—this remark will be important in our discussion of Berry's phase below.[12])

As a final check of our formalism we compute the equal-time correlation functions. Following the rules we have put forward, it is easy to show that

$$\begin{aligned}\rho(x,y) &\equiv \langle \Omega \mid \psi(x)\psi(y) \mid \Omega\rangle \\ &= \tfrac{1}{2}(I-\Omega)(x,y)\ .\end{aligned} \tag{3.72}$$

For the Hamiltonian (3.62), and the choice (3.69) for the vacuum, this is $\int (dp/2\pi)e^{-ip(x-y)}\theta(p)$, which is the conventional result.

In summary, let us contrast our fermionic Schrödinger picture with the familiar bosonic one. In both cases the functional space contains inequivalent Fock spaces. Choosing a specific quadratic Hamiltonian can select a specific Fock space for bosons, but not for fermions. In the former case, there is no sign ambiguity for the Gaussian covariance Ω because we require convergence of a Gaussian normalization integral, hence $\mathrm{Re}\Omega>0$; in the latter, the integral is Grassmannian, all integrals converge, and the sign of Ω is not fixed. Stated differently, a particle state is localized in the bosonic functional space, while there is no concept of localization in the Grassmann space. A unique fermionic Fock space requires prescribing a filling factor and restricting to properly projected polynomials in $\frac{1}{2}(I+\Omega)u$.

D. Anomalous commutators

The choice for filling the Dirac sea, which in our formalism corresponds to a choice for the covariance Ω of the vacuum state, affects the symmetry behavior of fermions, and gives rise to phenomena not present in the classical theory, for example, Schwinger terms in current commutators.

As an example we shall discuss within our formalism two-dimensional Weyl fermions, described by a Hamiltonian containing one-component, charged fermion fields:

$$H=\int \psi^\dagger h\psi\ . \tag{3.73}$$

As a consequence of the canonical anticommutator

$$\{\psi^\dagger(x),\psi(y)\}=\delta(x-y)\ , \tag{3.74}$$

the charge density

$$j=\tfrac{1}{2}[\psi^\dagger,\psi]\ , \tag{3.75}$$

defined through the commutator so that it is odd under charge conjugation ($\psi\leftrightarrow\psi^\dagger$), formally commutes with itself, even at different points; but this is false, owing to operator-product ambiguities. As soon as a definition of the Dirac sea is fixed by selecting Ω, the commutator acquires an anomalous c-number term.

Using the realization (3.28) for the operators ψ and $\psi^\dagger$ to solve the Hamiltonian eigenvalue problem, we find a Gaussian ground state,

$$|\Omega\rangle=\exp\left[\int u^\dagger\Omega u\right]\ , \tag{3.76a}$$

where as before the covariance satisfies (3.65) and

$$\Omega^2=I\ . \tag{3.76b}$$

Hence the normalization factor in (3.76a) is a phase, and has been ignored.

For a well-defined regularized charge density we take

$$j_F=\tfrac{1}{2}\int\left[u^\dagger(x)+\frac{\delta}{\delta u(x)}\right]F(x,y)\left[u(y)+\frac{\delta}{\delta u^\dagger(y)}\right]-\tfrac{1}{2}\operatorname{tr}F\ . \tag{3.77}$$

The symmetric Hermitian kernel $F(x,y)$ tends to an arbitrary diagonal kernel $f(x)\delta(x-y)$ in the local limit. The action of (3.77) on the vacuum is easily computed:

$$j_F\,|\,\Omega\rangle=\left[\tfrac{1}{2}\operatorname{tr}F\Omega+\tfrac{1}{2}\int u^\dagger(I-\Omega)F(I+\Omega)u\right]|\,\Omega\rangle\ . \tag{3.78}$$

In the same way one can compute the action of a second charge operator j_G on $j_F\,|\,\Omega\rangle$, and then evaluate the commutator of the two. The result is

$$[j_F,j_G]\,|\,\Omega\rangle=\left[\tfrac{1}{2}\operatorname{tr}[F,G]\Omega+\tfrac{1}{2}\int u^\dagger(I-\Omega)[F,G](I+\Omega)u\right]|\,\Omega\rangle\ . \tag{3.79}$$

In the local limit, the off-diagonal part vanishes, while the diagonal part, proportional to $|\,\Omega\rangle$, remains:

$$[j_f,j_g]\,|\,\Omega\rangle=-\tfrac{1}{2}\int dx\,dy\,f(x)g(y)\int\frac{dp}{2\pi}e^{-ip(x-y)}\int\frac{dq}{2\pi}\left[\Omega\left[q+\frac{p}{2}\right]-\Omega\left[q-\frac{p}{2}\right]\right]|\,\Omega\rangle\ . \tag{3.80}$$

We have taken the translation-invariant case, and $\Omega(p)$ is the Fourier-transformed covariance, antisymmetric with square one. Provided $\Omega(p)$ does not alternate between ±1 *ad infinitum*, the q integral converges. For an evaluation we write that integral as

$$\int\frac{dq}{2\pi}\int_{-1}^{1}dz\frac{p}{2}\Omega'\left[q+\frac{p}{2}z\right]\ .$$

$\Omega'(p)$ is a superposition of δ functions with alternating signs, concentrated at the finite number of points where $\Omega(p)$ jumps between -1 and $+1$:

$$\Omega'(p)=-2\sum_n\epsilon_n\delta(p-p_n),\quad \epsilon_n=\pm1\ . \tag{3.81}$$

Thus the q and z integrations leave $-cp/\pi$, where $c\equiv\sum_n\epsilon_n=\pm1$— the value depends on the direction of alterations, the total number of them being odd, since $\Omega(p)$ is an odd function. This gives finally

$$[j(x),j(y)]=\frac{ic}{2\pi}\delta'(x-y)\ . \tag{3.82}$$

Positivity of the energy spectrum requires $c=1$, which is the answer obtained with the conventional choice $\Omega(p)=\epsilon(-p)$, and coincides with the usual Schwinger term in the commutator of charge densities for Weyl fermions, whose current components coincide in the time and space directions.

Let us note that here the intrinsic method does not give definite results. The transformation kernel with the regularized generator (3.77) is of the same form as in (3.52), times an additional phase factor $e^{(i/2)\operatorname{tr}F}$ coming from the last term in (3.77). Also, F is now symmetric. Upon taking the local limit, the kernel in the Gaussian attains a well-defined expression

$$K_F(x,y)\to K_f(x,y)=\left[\cot\frac{f(x)}{2}\right]\delta(x-y)\ .$$

The normalization factor for imaginary τ is $e^{\operatorname{tr}(\ln\sinh F/2+F/2)}$. Because F is symmetric, tending to an arbitrary diagonal kernel in the local limit, there does not seem to be any well-defined way of extracting a unique finite part. Thus we see that the Schwinger term in the commutator of fermionic current densities is not determined intrinsically, but by the form of the Dirac sea—a fact established over a half-century ago.[13]

Finally, let us consider the commutator of conformal generators of Sec. III B by the same above method that gives the current commutator. For charged Weyl fermions, the form of the commutator anomaly is the same as (3.79) except that the kernels F and G now are the antisymmetric ones, appropriate to the regularized conformal transformations; see (3.32). We find in the local limit the extension

$$\tfrac{1}{2}\operatorname{tr}[F,G]\Omega=-\tfrac{1}{2}\int dx\,dy\,f(x)g(y)\int\frac{dp}{2\pi}e^{-ip(x-y)}\int\frac{dq}{2\pi}q^2\left[\Omega\left[q+\frac{p}{2}\right]-\Omega\left[q-\frac{p}{2}\right]\right]\ . \tag{3.83}$$

The q integral is rewritten as

$$\int \frac{dq}{2\pi} q^2 \int_{-1}^{1} dz \frac{p}{2} \Omega' \left| q + \frac{zp}{2} \right| ;$$

with Ω' as in (3.81), this leaves

$$-\frac{cp^3}{12\pi} - \frac{p}{\pi} \sum_n \epsilon_n p_n^2 ,$$

where as before $c \equiv \sum_n \epsilon_n$. Thus

$$\tfrac{1}{2}\operatorname{tr}[F,G]\Omega = -\frac{ic}{48\pi} \int (fg''' - gf''') - \frac{i}{4\pi} \sum_n \epsilon_n p_n^2 \int (fg' - gf') . \quad (3.84)$$

The second term is trivial, and may be reabsorbed in a redefinition of the conformal generators. The first agrees with (2.37), except now $c = \pm 1$, with the positive value required by positivity of the energy spectrum. This is reproduced when $\Omega(p) = \epsilon(-p)$.

For charged fermions one may also obtain $c > 1$ by using "improved" generators, $:Q_f: + \alpha \int f' j$. These effect a conformal transformation supplemented by a U(1) gauge transformation, $\delta_f \psi = (f\psi)' - (\frac{1}{2} + i\alpha) f' \psi$, and satisfy (2.37) with $c = 1 + 12\alpha^2 > 1$. The device of increasing the center by "improving" the generator is analogous to what can be done with bosons,[6] but is not applicable for the Majorana fermion, since there the improvement (j) vanishes. So we can represent conformal algebras with $c = \frac{1}{2}$ and $c \geq 1$, but thus far we have not found field-theoretic representations for the remaining discrete series between $\frac{1}{2}$ and 1 (Ref. 10).

E. Berry's phase and chiral anomalies

Berry showed that there is a correction to the quantum adiabatic theorem as stated in textbooks.[14] Consider a Hamiltonian $h(\mathbf{r})$ depending on parameters $\mathbf{r}(t)$, which evolve in time periodically and adiabatically: $\mathbf{r}(T) = \mathbf{r}(0)$. A state that coincides with an instantaneous eigenstate of h, $|n; \mathbf{r}(t)\rangle$, at $t = 0$ evolves as an instantaneous eigenstate for all times, but it acquires a phase, which after the cyclic evolution is $\exp[-i \int_0^T dt\, \epsilon_n(\mathbf{r}(t)) + i\gamma_n]$, where $\epsilon_n(\mathbf{r})$ is the instantaneous eigenvalue and γ_n is Berry's correction expressed as an integral over a connection $\mathcal{A}_n$:

$$\gamma_n = \oint d\mathbf{r} \cdot \mathcal{A}_n(\mathbf{r}) , \quad (3.85)$$

$$\mathcal{A}_n^i(\mathbf{r}) = i \left\langle n; \mathbf{r} \left| \frac{\partial}{\partial r^i} \right| n; \mathbf{r} \right\rangle . \quad (3.86)$$

An equivalent formula can be given in terms of the curvature

$$\mathcal{F}_n^{ij} = \frac{\partial}{\partial r^i} \mathcal{A}_n^j - \frac{\partial}{\partial r^j} \mathcal{A}_n^i \quad (3.87)$$

integrated over any surface enclosed by the adiabatically evolving closed path:

$$\gamma_n = \int d\Sigma^{ij} \mathcal{F}_n^{ij} . \quad (3.88)$$

The connection $\mathcal{A}^i$ undergoes a gauge transformation

$$\mathcal{A}_n^i \to \mathcal{A}_n^i - \frac{\partial}{\partial r^i} \Theta \quad (3.89)$$

when phases of the eigenstates are changed,

$$|n; \mathbf{r}\rangle \to e^{i\Theta} |n; \mathbf{r}\rangle , \quad (3.90)$$

but Berry's phase is gauge invariant. As an alternative to the formulas (3.86) and (3.87) involving variation of wave functions, the curvature $\mathcal{F}_n^{ij}$ may also be presented in terms of matrix elements of operators in the theory:

$$\mathcal{F}_n^{ij} = i \sum_{n' \neq n} \frac{\langle n; \mathbf{r} | h^i | n'; \mathbf{r}\rangle \langle n'; \mathbf{r} | h^j | n; \mathbf{r}\rangle}{(\epsilon_n - \epsilon_{n'})^2} - (i \leftrightarrow j) , \quad (3.91a)$$

$$h^i \equiv \frac{\partial}{\partial r^i} h . \quad (3.91b)$$

The above holds when the state $|n; \mathbf{r}\rangle$ is nondegenerate. In the case of degeneracy, one generalizes to a non-Abelian connection:

$$\mathcal{A}_{n;ab}^i = i \left\langle n, a; \mathbf{r} \left| \frac{\partial}{\partial r^i} \right| n, b; \mathbf{r} \right\rangle . \quad (3.92)$$

The non-Abelian curvature

$$\mathcal{F}_{n;ab}^{ij} = \frac{\partial}{\partial r^i} \mathcal{A}_{n;ab}^j - \frac{\partial}{\partial r^j} \mathcal{A}_{n;ab}^i - i[\mathcal{A}_n^i, \mathcal{A}_n^j]_{ab} \quad (3.93a)$$

may still be represented in terms of matrix elements of derivatives of h:

$$\mathcal{F}_{n;ab}^{ij} = i \sum_{c, n' \neq n} \frac{\langle n, a; \mathbf{r} | h^i | n', c; \mathbf{r}\rangle \langle n', c; \mathbf{r} | h^j | n, b; \mathbf{r}\rangle}{(\epsilon_n - \epsilon_{n'})^2} - (i \leftrightarrow j) . \quad (3.93b)$$

Here a, b, and c label the degeneracy.

It has further been shown that anomalous gauge theories, such as Weyl fermions in an external background gauge field, exhibit this phenomenon.[15] The analogy is drawn between the operators in h and the fermion field operators, the parameters $\mathbf{r}$ and the background gauge potential. The Fock states of the field theory—the analogs of $|n; \mathbf{r}\rangle$—depend on the background gauge potential, taken in the Weyl, $A^0 = 0$, gauge. Moreover, when the potentials are varied through a gauge transformation—this may be viewed as an adiabatic change—the Fock states of an anomalous theory acquire Berry's phase.

In the field-theory application with a nondegenerate ground state, $|\Omega; \mathbf{A}\rangle$, one is dealing with an Abelian connec-

tion, which is a functional of the background potential $\mathbf{A}$, with "components" labeled by i (spatial indices), a (group indices), and $\mathbf{x}$ (spatial coordinates):

$$\mathcal{A}_a^i(\mathbf{x};\mathbf{A})=i\left\langle\Omega;\mathbf{A}\left|\frac{\delta}{\delta A_a^i(\mathbf{x})}\right|\Omega;\mathbf{A}\right\rangle . \tag{3.94}$$

The Abelian functional curvature is

$$\mathcal{F}_{ab}^{ij}(\mathbf{x},\mathbf{y};\mathbf{A})\equiv\frac{\delta}{\delta A_a^i(\mathbf{x})}\mathcal{A}_b^j(y;\mathbf{A})-\frac{\delta}{\delta A_b^j(\mathbf{y})}\mathcal{A}_a^i(x;\mathbf{A}) \tag{3.95}$$

and may also be presented in terms of current matrix elements,

$$\mathcal{F}_{ab}^{ij}(\mathbf{x},\mathbf{y};\mathbf{A})=i\sum_{n\neq 0}\frac{1}{(\epsilon_n-\epsilon_0)^2}\langle\Omega;\mathbf{A}\,|\,J_a^i(\mathbf{x})\,|\,n;\mathbf{A}\rangle\langle n;\mathbf{A}\,|\,J_b^j(y)\,|\,\Omega;\mathbf{A}\rangle-(i,a,\mathbf{x}\leftrightarrow j,b,\mathbf{y})\ , \tag{3.96a}$$

the current being the variation of the Hamiltonian:

$$\frac{\delta H}{\delta\,\mathbf{A}_a(x)}=-\mathbf{J}_a(\mathbf{x})\ . \tag{3.96b}$$

It is interesting to study this effect in our formalism. However, before presenting field-theoretic results, we examine first a fermionic quantum-mechanical problem—Berry's paradigm for his phase.[14] This we do to clarify some of the issues that arise within our representation (3.4) for fermionic variables.

Let $h(\mathbf{r})$ be

$$h(\mathbf{r})=\tfrac{1}{2}\sigma\cdot\mathbf{r}\ . \tag{3.97}$$

The eigenvalues of (3.97) are $\pm r/2$, $r=|\,r\,|$, and the corresponding eigenvectors $|\pm;\mathbf{r}\rangle$ are readily constructed. From (3.86) and (3.87) one finds a Dirac monopole connection, with curvature

$$\mathcal{F}_{\pm}^{ij}=\mp\tfrac{1}{2}\epsilon^{ijk}\frac{\hat{r}^k}{r^2}\ . \tag{3.98}$$

This also follows directly from (3.91).

In order to apply our formalism to this problem we consider a Hamiltonian with three fermion variables $\psi(i)$:

$$h=\tfrac{1}{2}\psi(i)h_{ij}\psi(j)\ , \tag{3.99a}$$

$$h_{ij}=i\epsilon_{ikj}r^k(t)\ . \tag{3.99b}$$

That this coincides with Berry's model is recognized when the canonical anticommutator

$$\{\psi(i),\psi(j)\}=\delta_{ij} \tag{3.100}$$

is realized irreducibly:

$$\psi(i)=\frac{\sigma^i}{\sqrt{2}}\ . \tag{3.101}$$

With this two-dimensional representation for $\psi(i)$, (3.99) coincides with (3.97).

On the other hand, in our formalism $\psi(i)$ is given by

$$\psi(i)=\frac{1}{\sqrt{2}}\left[u(i)+\frac{\delta}{\delta u(i)}\right] \tag{3.102}$$

and operates on functions of three Grassmann variables $u(i)$. Evidently, this is an eight-dimensional space with basis: 1, $u^i\equiv u(i)$, $v^i\equiv\frac{1}{2}\epsilon^{ijk}u(j)u(k)$, and $w\equiv\frac{1}{6}\epsilon^{ijk}u(i)u(j)u(k)$; u^i and v^i are duals of each other and so are 1 and w. Therefore, there is a fourfold degeneracy in our description, which is reducible.

Notice that in contrast to our previous quantum-mechanical example in Sec. III A, here we are dealing with an odd number of variables, hence the "first quantized" Hamiltonian h_{ij} necessarily possesses a zero mode:

$$h_{ij}f_0^j=0\ , \tag{3.103a}$$

$$f_0^j=\hat{r}^j\ . \tag{3.103b}$$

The other first-quantized eigenvalues are $\pm r$:

$$h_{ij}f_{\pm}^j=\pm r f_{\pm}^i\ , \tag{3.104a}$$

$$f_{\pm}^i=\frac{1}{\sqrt{2}}(\hat{\theta}^i\pm i\hat{\phi}^i)\ . \tag{3.104b}$$

($\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ are spherical three-dimensional orthonormal vectors.) Because of the unpaired zero mode, this "field theory" behaves differently from what we have seen before. In fact, the example models vacuum degeneracy and fractional charge in continuum field theory, which arise when a zero mode is present.[12]

The fourfold degenerate eigenstate of (3.99) with negative eigenvalue $-r/2$, $|-,a;\mathbf{r}\rangle$, $a=1,2,3,4$, is represented by

$$|-,1;\mathbf{r}\rangle=\frac{1}{\sqrt{2}}(1+i\hat{\mathbf{r}}\cdot\mathbf{v})\ , \tag{3.105a}$$

$$|-,2;\mathbf{r}\rangle=\mathbf{f}_-\cdot\mathbf{v}\ , \tag{3.105b}$$

$$|-,3;\mathbf{r}\rangle=\frac{1}{\sqrt{2}}(w-i\hat{\mathbf{r}}\cdot\mathbf{u})\ , \tag{3.105c}$$

$$|-4;\mathbf{r}\rangle=\mathbf{f}_-\cdot\mathbf{u}\ . \tag{3.105d}$$

The four states with positive eigenvalue, $|+,a;\mathbf{r}\rangle$, are obtained from the above by application of the creation operator $\sum_i f_+^i\psi(i)$. [We do not use the previous Gaussian notation, which needs elaboration in the presence of zero modes. Two linear combinations in (3.105) are Gaussians; the other two are obtained by applying the

zero-mode operator $\sum_i \hat{\mathbf{r}}^i \psi(i)$ to Gaussians.]

Because of the degeneracy, Berry's connection is non-Abelian:

$$\mathcal{A}^i_{\pm ab} \equiv i\left\langle \pm,a;\mathbf{r}\left|\frac{\partial}{\partial r^i}\right|\pm,b;\mathbf{r}\right\rangle . \tag{3.106}$$

Nevertheless, the curvature $\mathcal{F}$, determined from $\mathcal{A}$ by the non-Abelian formula (3.93a) is Abelian:

$$\mathcal{F}^{ij}_{\pm} = \mp \tfrac{1}{2}\epsilon^{ijk}\frac{\hat{r}^k}{r^2}\begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} . \tag{3.107}$$

This rederives (3.98) within our formalism, and shows the Berry's connection and curvature belong to a U(1) group, since the degeneracy is an artifact of our representation. Therefore, there is a gauge—a linear combination of states (3.105)—such that the matrix connection (3.106) factorizes into a function times a four-dimensional unit matrix, putting into evidence the essentially Abelian nature of the connection. This may be verified by explicit and lengthy computation. But is is easy to circumvent the unnecessary complications of the artifactual non-Abelian structure, by calculating the curvature directly from (3.93b), which involves only matrix elements of operators and not variations of states. (It is the variation of the state which connects to the degenerate copy spaces. This is avoided by varying operators. See the discussion at the end of Sec. III C above.) The computation from (3.93b) is especially simple, because the sum involves only one state and thus $(\epsilon_n - \epsilon_{n'})^2 = r^2$. Agreement with (3.98) is readily obtained.

We now turn to the field-theoretic example. From the above quantum mechanical exercise we know that the degeneracy of the representation gives rise to a connection which appears non-Abelian in the space of degenerate states. On the other hand the curvature is Abelian. The formula for the connection (3.94) involves variation with respect to parameters—the vector potentials. This in general takes one out of the Fock space of the theory, by producing expressions that depend not only on the projected Grassmann field u_+, but also on the complementary u_-. On the other hand, formula (3.96) for the curvature involves matrix elements of the current operator $\mathbf{J} \equiv -\delta H/\delta \mathbf{A}$. This does not take one out of a definite Fock space.

We consider two-dimensional Weyl fermions (one component complex spinors) interacting with an external electromagnetic gauge field, which in the Weyl ($A^0=0$) gauge possesses one component A. If the first-quantized Hamiltonian without electromagnetism is $h(x,y)$, the Hamiltonian for the field theory is

$$H = \int \psi^\dagger h_A \psi , \tag{3.108}$$

$$h_A(x,y) = \exp\left[-ie\int_y^x A(z)dz\right]h(x,y) . \tag{3.109}$$

[For the familiar case (3.62), $h_A(x,y) = i\delta'(x-y) - eA(x)\delta(x-y)$.] With the representation (3.28) for the charged fermion fields, the ground state is Gaussian,

$$|\Omega_A\rangle = \exp\left[\int u^\dagger \Omega_A u\right] \tag{3.110}$$

with the covariance Ω_A satisfying the usual equation

$$(I-\Omega_A)h_A(I+\Omega_A) = 0 . \tag{3.111}$$

The solution to (3.111) is presented in terms of the appropriate covariance in the absence of interaction. If Ω satisfies

$$(I-\Omega)h(I+\Omega) = 0 \tag{3.112}$$

then (3.111) is solved by

$$\Omega_A(x,y) = \exp\left[-ie\int_y^x dz\, A(z)\right]\Omega(x,y) \tag{3.113}$$

and it also follows that $\Omega_A^2 = I$, since $\Omega^2 = I$. Therefore, the ground-state wave functional is

$$|\Omega;A\rangle = N(A)\exp\left[\int dx\,dy\,u^\dagger(x)\exp\left[-ie\int_y^x dz\,A(z)\right]\Omega(x,y)u(y)\right] . \tag{3.114}$$

Since $\Omega_A = \Omega_A^\dagger = \Omega_A^{-1}$, the normalization factor $N(A)$ is a phase whose form we cannot determine but which can depend on A.

The computation of Berry's phase now proceeds as follows. From (3.114) we have

$$i\frac{\delta}{\delta A(x)}|\Omega;A\rangle = \left[i\frac{\delta}{\delta A(x)}\ln N(A) + e\int dy\,dz\,u^\dagger(y)\Omega_A(y,z)[\theta(y-z)\theta(y-x)\theta(x-z) - (y\leftrightarrow z)]u(z)\right]|\Omega;A\rangle . \tag{3.115}$$

The second term, being off-diagonal, does not contribute to $\langle\Omega;A|i\delta/\delta A(x)|\Omega;A\rangle$, which therefore is $[i\delta/\delta A(x)]\ln N(A)$. This however does not mean that Berry's connection is a pure (functional) gauge, with vanishing (functional) curvature. Rather we recall from the quantum-mechanical example that the reducibility of our representation gives rise to degeneracy in our states and consequently a non-Abelian connection, even though the curvature is ultimately Abelian. More specifically, we see from (3.115) that the effect of $\delta/\delta A$ on $|\Omega;A\rangle$ produces terms quadratic in u, where both projections u_+ and u_- occur. However, we have no effective way of labeling the degeneracy and calculating the non-Abelian connection.

Fortunately, the above obstacle may be circumvented by proceeding directly to the curvature as given by (3.96). The sum over the intermediate states now extends over two-particle states; thus, for the case $h(x,y)=i\delta'(x-y)$,

$$\mathcal{F}(x,y;A)=e^2 i\int_0^\infty \frac{dk\,dp}{(k+p)^2}[\langle\Omega;A\mid\psi^\dagger(x)\psi(x)\mid k,p\rangle\langle k,p\mid\psi^\dagger(y)\psi(y)\mid\Omega;A\rangle-(x\leftrightarrow y)] \ . \tag{3.116}$$

This gives

$$\mathcal{F}(x,y;A)=-\frac{e^2}{4\pi}\epsilon(x-y) \tag{3.117}$$

in agreement with previous evaluations.[15]

Please observe that apart from the (unknown) normalization, $N(A)$, the vacuum is gauge invariant, in the sense that a gauge transformation $A\to A+\theta'$ is compensated by $u\to e^{-ie\theta}u$. The anomalous Berry phase arises from the fact that an adiabatic change in A takes one out of the selected Fock space. This may also be seen by operating on $\mid\Omega;A\rangle$ with the Gauss-law generator $\mathcal{G}(x)$:

$$\mathcal{G}(x)=\frac{1}{ie}\frac{d}{dx}\frac{\delta}{\delta A(x)}-\tfrac{1}{2}[\psi^\dagger(x),\psi(x)] \ . \tag{3.118}$$

The effect of each of the two terms on $\mid\Omega;A\rangle$ is

$$\frac{1}{i}\frac{d}{dx}\frac{\delta}{\delta A(x)}\mid\Omega;A\rangle=\left[\frac{1}{ie}\frac{d}{dx}\frac{\delta}{\delta A(x)}\ln N(A)+\int dy[u^\dagger(x)\Omega_A(x,y)u(y)-u^\dagger(y)\Omega_A(y,x)u(x)]\right]\mid\Omega;A\rangle \ , \tag{3.119a}$$

$$\tfrac{1}{2}[\psi^\dagger(x),\psi(x)]\mid\Omega;A\rangle=\tfrac{1}{2}\int dy\,dz\,u^\dagger(y)[\delta(y-x)-\Omega_A(y,x)][\delta(x-z)+\Omega_A(x,z)]u(z)\mid\Omega;A\rangle \ . \tag{3.119b}$$

The first term involves variation with respect to parameters and gives rise to both physical and unphysical projections of u and $u^\dagger$. When combined with the second term, there remain only the unphysical projections $u^\dagger(I+\Omega)$ and $(I-\Omega)u$:

$$\mathcal{G}(x)\mid\Omega;A\rangle=\left[\frac{1}{ie}\frac{d}{dx}\frac{\delta}{\delta A(x)}\ln N(A)-\tfrac{1}{2}\int dy\,dz\,u^\dagger(y)[\delta(y-x)+\Omega(y,x)][\delta(x-z)-\Omega(x,z)]u(z)\right]\mid\Omega;A\rangle \ . \tag{3.120}$$

Therefore in our formalism, the Gauss generator fails to leave the vacuum invariant by taking that state out of the properly projected Fock space.

Also we may operate with a second Gauss generator, $\mathcal{G}(y)$, on (3.120). By taking the antisymmetric combination and regularizing, the expected anomalous commutator is found:

$$[\mathcal{G}(x),\mathcal{G}(y)]=\frac{i}{2\pi}\delta'(x-y) \ . \tag{3.121}$$

IV. CONCLUSION

We have constructed a Grassmann functional representation for fermion quantum fields, which is analogous to the familiar Schrödinger picture for boson quantum fields. We employed a reducible realization for the infinite, field-theoretic Clifford algebra in terms of Grassmann variables. Within the formalism we found a projective representation for the two-dimensional conformal group, with central extension in its Lie algebra $c=\frac{1}{2}$ and $c\geq 1$, whose form coincides with that of the bosonic representation, except for variations arising from differences between the bosonic and fermionic canonical structures.

The reducibility of the representation gives rise to inequivalent Fock bases in the functional space, even when a specific Hamiltonian dynamics is selected. This allows us to survey the manifold of all Fock spaces.[16] For quadratic Hamiltonians, we determined the Fock eigenstates, and were able to identify the peculiarities of Fermionic quantum field theory: Dirac sea, fractional charge, Berry's phase, and the anomaly phenomenon. While none of these results are new, our formalism provides a fresh point of view, in terms of explicit formulas. For example, the anomaly phenomenon was seen to arise because the Gauss generator takes the Fock vacuum into an inequivalent Fock state.

We envision further applications of our formalism. The representation for discrete series of conformal transformations[10] remains an open problem. We wish to understand two-dimensional boson-fermion equivalence in this framework. Supersymmetric theories, in particular representations of the two-dimensional superconformal group, are being studied.[17]

While only linear dynamics was considered in this paper, it should be possible to analyze simple nonlinear theories, such as the Thirring model. Systems with nontrivial interactions can be studied by variational methods, where explicit trial wave functionals can now be constructed for fermions as well as bosons.[18]

Note added in proof. We have learned that a reducible representation of the Clifford algebra, similar to ours, has appeared in the mathematics literature [L. Gross, J. Funct. Anal. **25**, 162 (1977).]

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APPENDIX

There are other functional representations for fermionic quantum fields in terms of Grassmann fields. They differ from ours in that they provide an irreducible representation of the (infinite) Clifford algebra (3.3), in contrast with our reducible representation. This is achieved by a variety of splittings of the quantum field ψ, as explained in the main body of our paper.

In the holomorphic representation, described, for example, by Faddeev and Slavnov,[19] the splitting is achieved by decomposing ψ into annihilation and creation operators and representing the former by Grassmann multiplication and the latter by functional Grassmann differentiation.

The splitting may be performed in position space when the fermion operator possesses more than one component. Thus for charged fermion fields ψ, composed of two Majorana fields ψ_1 and ψ_2, one may represent ψ_1 by $(1/\sqrt{2})(u+\delta/\delta u)$ and ψ_2 by $(1/\sqrt{2}i)(u-\delta/\delta u)$, giving $\psi\equiv(1/\sqrt{2})(\psi_1+i\psi_2)$ as u and $\psi^\dagger$ as $\delta/\delta u$. This is the choice made by Barnes and Ghandour[8] who also introduce the dual space (3.9). Baaquie[20] adopts a related approach to spinors that possess an even number of components, which are split into two sets and are represented irreducibly.

Clearly neither of the last two methods is applicable for a one-component fermion, while the holomorphic representation is unavailable if one avoids choosing a creation-annihilation decomposition.

A representation identical to ours for fermionic quantum mechanics is described by DeWitt.[21] But the inner product is quite different; it is Grassmann valued.

To elaborate further on our definition for the dual of a vector introduced in Sec. III, Eqs. (3.12) and (3.20), we show a that a similar definition can also be used in a more familiar case: the Hilbert space of square-integrable functions on the infinite line.

A complete basis for the space is given by harmonic-oscillator wave functions, with arbitrary frequency ω,

$$|n\rangle \leftrightarrow u_n(x)=\left[\frac{\sqrt{\omega}}{\sqrt{\pi}\,2^n n!}\right]^{1/2} e^{-\omega x^2/2} H_n(\sqrt{\omega}\,x)\ , \qquad \text{(A1)}$$

where H_n are Hermite polynomials. The vectors $|n\rangle$ are orthonormal with respect to the standard inner product:

$$\langle f\,|\,g\rangle \equiv \int dx\, f^*(x) g(x)\ , \qquad \text{(A2)}$$

$$\langle m\,|\,n\rangle = \int dx\, u_m(x) u_n(x)=\delta_{mn}\ . \qquad \text{(A3)}$$

In this case, the dual basis vectors, represented by the complex-conjugated functions, can be identified with the original basis, and the space coincides with the dual space.

Out of this basis, a different complete set of vectors $|\tilde{n}\rangle$ for the Hilbert space can be defined; the new states are still orthonormal but with respect to a different scalar product. We choose

$$|\tilde{n}\rangle \leftrightarrow v_n(x)=\left[\frac{1}{\sqrt{2\omega}}\right]^{n+1/2} e^{\omega x^2/2} u_n(x) = \left[\frac{1}{\sqrt{2\pi}\,\omega^n 2^{2n} n!}\right]^{1/2} H_n(\sqrt{\omega}\,x) \qquad \text{(A4)}$$

and define the inner product $\langle \tilde{m}\,|\,\tilde{n}\rangle$ to be the expression given in (A3), which can also be written as

$$\langle \tilde{m}\,|\,\tilde{n}\rangle = \int dx\, v_m^\dagger(x) v_n(x) \qquad \text{(A5)}$$

provided the dual of $|\tilde{n}\rangle$ is represented by

$$\langle \tilde{n}\,| \leftrightarrow v_n^\dagger(x)=(\sqrt{2\omega})^{n+1/2} e^{-\omega x^2/2} u_n(x) = \left[\left[\frac{2}{\pi}\right]^{1/2} \frac{\omega^{n+1}}{n!}\right]^{1/2} e^{-\omega x^2} H_n(\sqrt{\omega}\,x)\ . \qquad \text{(A6)}$$

Using the generating formula for the Hermite polynomials,

$$H_n(z)=(-1)^n e^{z^2}\frac{d^n}{dz^n}e^{-z^2}\ ,$$

one can rewrite (A6) as

$$v_n^\dagger(x)=(-1)^n\left[\frac{\sqrt{2\pi}}{n!}\right]^{1/2}\frac{d^n}{dx^n}\left[\left[\frac{\omega}{\pi}\right]^{1/2} e^{-\omega x^2}\right]\ . \qquad \text{(A7)}$$

Since the above is a basis and a dual basis for any value of the frequency, we may let ω pass to infinity. The representation of $|\tilde{n}\rangle$ then becomes

$$|\tilde{n}\rangle \leftrightarrow v_n(x)=\frac{x^n}{(\sqrt{2\pi}\,n!)^{1/2}} \qquad \text{(A8)}$$

with the dual given by

$$\langle \tilde{n}\,| \leftrightarrow v_n^\dagger(x)=(-1)^n\left[\frac{\sqrt{2\pi}}{n!}\right]^{1/2}\frac{d^n}{dx^n}\delta(x)\ . \qquad \text{(A9)}$$

These are the one-dimensional bosonic analogues of the definitions used in Sec. III.

In particular, with (A8) and (A9) one can check that the dual of a Gaussian ($\mathrm{Re}\Omega<0$, $|\Omega|<1$)

$$|\Omega\rangle \leftrightarrow \Psi_\Omega(x)=\Omega^{1/4} e^{x\Omega x/2} \qquad \text{(A10)}$$

is

$$\langle \Omega\,| \leftrightarrow \Psi_\Omega^\dagger(x)=(\Omega^{*-1})^{1/4} e^{-x(\Omega^{*-1})x/2} \qquad \text{(A11)}$$

with

$$\langle \Omega\,|\,\Omega\rangle = \left[\frac{1}{2\pi}[(\Omega\Omega^*)^{-1/2}-(\Omega\Omega^*)^{1/2}]\right]^{-1/2}\ . \qquad \text{(A12)}$$

The analogy with (3.14), (3.15), and (3.16) is apparent.

[1]Our previous work on representation theory in the functional Schrödinger picture is contained in R. Floreanini and R. Jackiw, Phys. Lett. B **175**, 428 (1986); R. Floreanini, C. Hill, and R. Jackiw, Ann. Phys. (N.Y.) **175**, 354 (1987); R. Floreanini, *ibid.* **178**, 227 (1987); R. Floreanini and L. Vinet, Phys. Rev. D **36**, 1731 (1987). For reviews, see R. Jackiw, in *Super Field Theories*, edited by H. Lee, V. Elias, G. Kunstatter, R. Mann, and K. Viswanathan (Plenum, New York, 1987); in *Conformal Field Theory, Anomalies and Superstrings*, edited by B. Baaquie, C. Chew, C. Oh, and K. K. Phua (World Scientific, Singapore, 1988); R. Floreanini and L. Vinet, in *Proceedings of the CAP-NSERC Summer Institute of Theoretical Physics*, Edmonton, 1987, edited by F. Khanna, H. Umezawa, G. Kunstatter, and H. Lee (World Scientific, Singapore, 1988).

[2]In the Appendix we compare our functional approach to fermion fields with others.

[3]For a review, see R. Jackiw, in *Progress in Quantum Field Theory*, edited by H. Ezawa and S. Kamefuchi (North-Holland, Amsterdam, 1986).

[4]Floreanini, Hill, and Jackiw (Ref. 1); Floreanini and Vinet (Ref. 1).

[5]A. Niemi and G. Semenoff, Phys. Lett. B **176**, 108 (1986); H. Neuberger, A. Niemi, and G. Semenoff, *ibid.* **181**, 244 (1986).

[6]Representations with $c \geq 1$ arise from "improved" generators $:Q_f:+(\alpha/\sqrt{2\pi})\int f'\chi$. These effect an inhomogeneous transformation, $\delta_f\chi=(\chi f)'+(\alpha/\sqrt{2\pi})f''$, and satisfy (2.37) with $c=1+12\alpha^2>1$. For details, see Floreanini (Ref. 1).

[7]In this matter, see the recent investigation by J. Mickelsson and S. Rajeev, Comm. Math. Phys. (to be published).

[8]This definition of the dual space was given by T. Barnes and G. Ghandour, Nucl. Phys. **B146**, 483 (1978); Czech. J. Phys. B **29**, 256 (1979); see Appendix.

[9]The analogy is brought out as follows. On a two-dimensional space one can define the zero-form, two one-forms dx^μ, $\mu=1,2$, and an area two-form $\frac{1}{2}\epsilon_{\mu\nu}dx^\mu dx^\nu$. These are analogs to the four basis elements in (3.6). The duals of the forms are, respectively, the area two-form, the two one-forms $\epsilon_{\mu\nu}dx^\nu$, and the zero-form. These are analogs of the four basis elements in the dual space (3.9).

[10]D. Friedan and S. Shenker, in *Unified String Theories*, edited by M. Green and D. Gross (World Scientific, Singapore, 1986).

[11]S. Fubini, G. Hanson, and R. Jackiw, Phys. Rev. D **7**, 1732 (1973). With charged fermions, one may also achieve $c>1$. This is explained below.

[12]When h possesses an isolated zero mode, Ω is undetermined in that channel. This is the manifestation in our formalism of the vacuum degeneracy and fractional charge phenomenon; for a review, see A. Niemi and G. Semenoff, Phys. Rep. **135**, 99 (1986).

[13]The first observations on this subject are by P. Jordan in the 1930s and are described by C.-H. Tze, in *Chiral Solitons*, edited by K. Liu (World Scientific, Singapore, 1987).

[14]M. V. Berry, Proc. R. Soc. London **A392**, 45 (1984). For a review, see R. Jackiw, Comments At. Mol. Phys. **21**, 71 (1988).

[15]For a review, see G. Semenoff, in *Super Field Theories* (Ref. 1); A. Niemi, in *Workshop on Skyrmions and Anomalies*, edited by M. Jezabek and M. Praszalowicz (World Scientific, Singapore, 1987).

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[19]L. Faddeev and A. Slavnov, *Gauge Fields* (Benjamin/Cummings, Menlo Park, CA, 1980).

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[21]B. DeWitt, *Supermanifolds* (Cambridge University Press, Cambridge, England, 1984).